

HOCHMAN'S UPCROSSING THEOREM FOR GROUPS OF POLYNOMIAL GROWTH.

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ABSTRACT. Consider a stochastic process $(S_{[a_i, b_i]})_{[a_i, b_i] \subset \mathbb{N}}$, which is indexed by the collection of all nonempty intervals $[a_i, b_i] \subset \mathbb{N}$ and which is stationary under translations of the intervals. It was shown by M. Hochman that, for any $k \geq 1$ and any interval $(\alpha, \beta) \subset \mathbb{R}$, one can give an ‘almost-exponential’ bound on the size of the set where the associated process $(S_{[1, n]})_{n \geq 1}$ has at least k fluctuations over (α, β) . It was also noticed that a similar techniques can be applied in \mathbb{Z}^d case. In this article we extend Hochman’s upcrossing theorem to groups of polynomial growth.

1. INTRODUCTION

Given an integer $n \in \mathbb{Z}_{\geq 0}$ and some numbers $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \beta$, a sequence of real numbers $(a_i)_{i=1}^k$ is said to have **at least n upcrossings** across the interval (α, β) if there are indexes $1 \leq i_1 < j_1 < i_2 < j_2 < \dots < i_n < j_n \leq k$ such that

- 1) $a_{i_l} < \alpha$ for all indexes l ;
- 2) $a_{j_l} > \beta$ for all indexes l .

If $(a_i)_{i \geq 1}$ is an infinite sequence of real numbers, we use the same terminology and say that $(a_i)_{i \geq 1}$ has at least n upcrossings across the interval (α, β) if some initial segment $(a_i)_{i=1}^k$ of the sequence has at least n upcrossings across (α, β) . To simplify the notation we denote the sets of real-valued sequences having at least n upcrossings across an interval (α, β) by $\mathcal{F}_{(\alpha, \beta)}^n$.

The purpose of this article is to generalize the following result of M. Hochman from [Hoc09]. Consider a real-valued stochastic process $(S_{[i, j]})_{1 \leq i \leq j}$, indexed by the collection of all integer intervals $[i, j] \subset \mathbb{N}$. Suppose that this process is stationary, i.e.,

$$(S_{[i, j]})_{1 \leq i \leq j} = (S_{[i+k, j+k]})_{1 \leq i \leq j} \quad \text{for all } k \in \mathbb{N}$$

in distribution. For a measure space (Y, \mathcal{C}, ν) , a measurable subset $I \subseteq Y$ and a number $\delta > 0$ we say that a collection of measurable sets I_1, \dots, I_k **δ -fills** I if $I_i \subseteq I$ for all indices i and $\nu\left(I \setminus \bigcup_i I_i\right) < \delta\nu(I)$. When working with subsets of \mathbb{N} , or subsets of groups of polynomial growth, we always use the counting measure. The result of Hochman states the following. Let $(\alpha, \beta) \subset \mathbb{R}$ be some interval and $0 < \delta < \frac{1}{4}$ be some constant. Then there exist constants $c > 0$ and $\rho \in (0, 1)$, depending only on α, β, δ , such that for every stationary process $(S_{[i, j]})_{1 \leq i \leq j}$ and all $k \geq 1$

$$\begin{aligned} & \mathbb{P}(\{x : (S_{[1, i]}(x))_{i \geq 1} \in \mathcal{F}_{(\alpha, \beta)}^k\}) \leq \\ & \leq c\rho^k + \mathbb{P}\left(\left\{x : \begin{array}{l} \exists n > k \text{ s.t. } S_{[1, n]}(x) > \beta \text{ and } B(n) \text{ can be } \delta\text{-filled} \\ \text{by disjoint intervals } V_1, \dots, V_m \text{ s.t. } \forall i \ S_{V_i}(x) < \alpha \end{array} \right\}\right). \end{aligned}$$

Date: December 19, 2016.

2000 Mathematics Subject Classification. Primary 28D15, 60G10, 60G17.

We generalize this result as follows. Suppose that Γ is an discrete group of polynomial growth endowed with the word metric d , see Section 2.2 for the definitions. We consider stationary processes $(S_{B(g,r)})$, indexed by the collection of all balls

$$\{B(g, r) : g \in \Gamma, r \in \mathbb{Z}_{\geq 0}\},$$

which we view plainly as subsets of Γ for the moment. We denote by $B(n)$ the ball $B(e, n)$ of radius n around the neutral element $e \in \Gamma$. Stationarity of the process means that

$$(S_{B(g,r)})_{g,r} = (S_{B(g+h,r)})_{g,r} \quad \text{for all } h \in \Gamma$$

in distribution. The main result of this article is the following theorem.

Theorem 1.1. *Let Γ be a group of polynomial growth. For all intervals $(\alpha, \beta) \subset \mathbb{R}$ and all $\delta > 0$ small enough there exist constants $c > 0$ and $\rho \in (0, 1)$ such that the following holds. If $(S_{B(g,r)})$ is a stationary process, then for all $k \geq 1$*

$$\begin{aligned} & \mathbb{P}(\{x : (S_{B(i)}(x))_{i \geq 1} \in \mathcal{F}_{(\alpha, \beta)}^k\}) \leq \\ & \leq c\rho^k + \mathbb{P}\left(\left\{x : \begin{array}{l} \exists n > k \text{ s.t. } S_{B(n)}(x) > \beta \text{ and } B(n) \text{ can be } \delta\text{-filled} \\ \text{by disjoint balls } V_1, \dots, V_m \text{ s.t. } \forall i \ S_{V_i}(x) < \alpha \end{array} \right\}\right) \end{aligned}$$

The paper is structured as follows. In Section 2.1 we introduce the notion of a metric measure space of exact polynomial volume growth, provide some examples and prove some elementary properties of these spaces. Section 2.2 is devoted to discrete groups of polynomial growth, which become metric measure spaces of uniformly exact polynomial volume growth when endowed with the counting measure and the word metric. In Section 3 we derive some covering lemmas on metric measure spaces of exact polynomial volume growth, including the ‘Effective Vitali Covering’ (Theorem 3.5) and the generalization of Hochman’s ‘tower sandwich’ lemma (Theorem 3.8), which are the main technical tools in this article.

The upcrossing theorem for groups of polynomial growth is proved in Section 4.1. As one of the applications, we show in Section 4.2 how Hochman’s proof of exponential upcrossing inequality for Kingman’s subadditive ergodic theorem can be obtained for discrete groups of polynomial growth using Theorem 1.1.

2. PRELIMINARIES

2.1. Metric Measure Spaces of Exact Polynomial Volume Growth. Let $X = (X, d)$ be a metric space. A closed ball with radius $r \in \mathbb{R}_{\geq 0}$ and center $x \in X$ will be denoted by $B(x, r)$. Let μ be a Borel measure on X . A pair (X, μ) will be called a **metric measure space**. We say that the measure μ on X is **strictly positive** if for all points $x \in X$ and all radii $r \in \mathbb{R}_{> 0}$

$$0 < \mu(B(x, r)) < \infty.$$

A metric measure space (X, μ) with a strictly positive measure μ will be called a **space of exact polynomial volume growth** if there are constants $c_X \in \mathbb{R}_{> 0}$ and $q \in \mathbb{R}_{\geq 0}$ such that

$$(2.1) \quad \lim_{t \rightarrow \infty} \frac{\mu(B(x, t))}{c_X t^q} = 1 \quad \text{for all } x \in X.$$

The fact that μ is strictly positive implies that the constants c_X and q are uniquely determined. We call q the **degree of polynomial volume growth**. If the limit in Equation (2.1) converges uniformly on a subset $W \subseteq X$, we say that (X, μ) has **uniformly exact polynomial volume growth on W** . If the limit converges uniformly on the whole space X , the space (X, μ) will be called a **space of uniformly exact polynomial volume growth**.

Many interesting examples of metric measure spaces with (uniformly) exact polynomial volume growth are given by groups endowed with a translation-invariant metric and a Haar measure. We begin with some basic examples.

Example 2.1. Let $X := (\mathbb{R}^k, d)$ be the k -dimensional Euclidean space, which we endow with the ℓ^2 -norm $\|\cdot\|_2$ and the associated translation-invariant metric d given by

$$d(x, y) := \|x - y\|_2.$$

It is easy to see that the Lebesgue measure μ on \mathbb{R}^k is doubling, and, furthermore, that (X, μ) is a metric measure space of uniformly exact polynomial volume growth of degree k .

Example 2.2. Consider the real Heisenberg group $\mathrm{UT}_3(\mathbb{R})$. By definition,

$$\mathrm{UT}_3(\mathbb{R}) := \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

To simplify the notation, we will denote a matrix

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{UT}_3(\mathbb{R})$$

by the corresponding triple (a, b, c) of its entries. The Lebesgue measure on \mathbb{R}^3 becomes the Haar measure μ on $\mathrm{UT}_3(\mathbb{R})$ under this identification, i.e., for all compactly supported continuous functions $f \in C_c(\mathrm{UT}_3(\mathbb{R}))$ we have

$$\int_{\mathrm{UT}_3(\mathbb{R})} f d\mu = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(a, b, c) da db dc.$$

The *Korányi norm* of an element $(a, b, c) \in \mathrm{UT}_3(\mathbb{R})$ is defined by

$$\|(a, b, c)\|_K := ((a^2 + b^2)^2 + c^2)^{1/4},$$

and the associated left-invariant metric is given by

$$d_K(x, y) := \|x^{-1}y\|_K \quad (x, y \in \mathrm{UT}_3(\mathbb{R})).$$

A direct computation shows that, with respect to this metric, $((\mathrm{UT}_3(\mathbb{R}), d_K), \mu)$ is a metric measure space of uniformly exact polynomial volume growth of degree 4.

Remark 2.3. Furthermore, one can generalize these examples and show that connected nilpotent homogeneous groups have uniformly exact polynomial volume growth as well. For more details we refer to [Nev06]. For the definition of a homogeneous group we refer to [FS82]. We will not discuss these results in the article, since our main object of interest in this paper are (discrete) *groups of polynomial growth*, which become metric measure spaces with uniformly exact polynomial volume growth when endowed with a counting measure and a *word metric*. We introduce groups of polynomial growth in the next section (Section 2.2).

Remark 2.4. A metric measure space (X, μ) with a strictly positive measure μ is called **doubling** if there exists a constant $C > 0$ such that for all $x \in X, r \in \mathbb{R}_{\geq 0}$ we have

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)).$$

The concept of a doubling metric measure space is standard in analysis. One may wonder if a metric measure space of uniformly exact polynomial growth is necessarily doubling - and the answer in general is ‘no’. The reason is that the definition of uniformly exact polynomial growth only controls the size of the balls ‘at large scale’, while the doubling condition at small scales can still be violated.

A natural question is if all metric measure spaces of exact polynomial volume growth have uniformly exact polynomial volume growth. The answer is ‘no’: it is easy to construct a metric d' on \mathbb{R} such that

- (a) d' is equivalent to d in the sense of topology (but *not* bi-Lipschitz equivalent);

- (b) $((\mathbb{R}, d'), \mu)$ is a metric measure space of exact, but *not* uniformly exact, polynomial growth of degree 1.

However, the following lemma shows that the volume growth is always uniformly exact on bounded subsets of metric measure spaces of exact polynomial growth.

Lemma 2.5. *Let (X, μ) be a metric measure space of exact polynomial growth. Let $W = B(x, r) \subseteq X$ be a ball around $x \in X$. Then (X, μ) has uniformly exact polynomial growth on W .*

Proof. Fix a point $y \in W$. Then the proof easily follows from the definition of exact polynomial growth combined with the observations that

$$B(x, s - r) \subseteq B(y, s) \quad \text{for } s > r$$

and

$$B(y, s) \subseteq B(x, r + s) \quad \text{for } s > r.$$

□

If $t \in \mathbb{R}_{\geq 0}$ is some number and $W = B(y, s) \subseteq X$ is some ball, we will denote by $t \cdot W$ the ball $B(y, ts)$. For a ball $W = B(y, s)$ in a metric measure space (X, μ) and a number $\delta \in (0, 1)$ we define the **δ -interior** of W as the ball

$$\text{int}_\delta(W) := B(y, (1 - \delta)s),$$

and the **δ -boundary** of W as the set

$$\partial_\delta(W) := W \setminus \text{int}_\delta(W).$$

For $\delta \in \mathbb{R}_{\geq 0}$ the **δ -expansion** of W is the ball

$$W^\delta := B(y, (1 + \delta)s).$$

In principle, these objects do depend on a concrete representation of W as a ball $B(y, s)$, and not just on W as a subset of X . Thus, whenever we are talking about balls or collections of balls, we always assume that concrete centers and radii are provided, and this will always be the case in applications. Given a collection \mathcal{C} of balls in (X, μ) and a number $\delta \in (0, 1)$, we define the **δ -interior** of \mathcal{C} as

$$\text{int}_\delta(\mathcal{C}) := \bigcup_{W \in \mathcal{C}} \text{int}_\delta(W),$$

and the **δ -boundary** of \mathcal{C} as

$$\partial_\delta(\mathcal{C}) := \bigcup_{W \in \mathcal{C}} \partial_\delta(W).$$

For $\delta \in \mathbb{R}_{\geq 0}$ the **δ -expansion** of \mathcal{C} is

$$\mathcal{C}^\delta := \bigcup_{W \in \mathcal{C}} W^\delta.$$

Next, the **radius** of \mathcal{C} is defined as the infimum of radii of balls in \mathcal{C} , i.e.,

$$\text{rad}(\mathcal{C}) := \inf\{r : B(x, r) \in \mathcal{C}\},$$

and the **core** of \mathcal{C} is defined as the set of centers of balls in \mathcal{C} , i.e.,

$$\text{core}(\mathcal{C}) := \{x : B(x, r) \in \mathcal{C}\}.$$

We will now prove a basic lemma, saying that in metric measure spaces of exact polynomial volume growth we can control the size of the δ -boundary of collections of balls with sufficiently large radius.

Lemma 2.6 ('Controlling the size of δ -boundary'). *Let (X, μ) be a metric measure space of exact polynomial growth of degree $q > 0$ and $X_0 \subseteq X$ be a bounded set. There exists a number $\varepsilon_0 \in (0, 1)$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there exists a number $s = s(\varepsilon, X_0)$ so that the following assertion holds. If \mathcal{C} is a countable collection of disjoint balls in X such that*

$$\text{rad}(\mathcal{C}) \geq s$$

and

$$\text{core}(\mathcal{C}) \subseteq X_0,$$

then for all $0 < \delta \leq \min(\frac{\varepsilon}{4q}, \frac{1}{2})$ we have

$$(2.2) \quad \mu(\text{int}_\delta(\mathcal{C})) \geq (1 - \varepsilon) \mu \left(\bigsqcup_{U \in \mathcal{C}} U \right)$$

and

$$(2.3) \quad \mu(\partial_\delta(\mathcal{C})) \leq \varepsilon \mu \left(\bigsqcup_{U \in \mathcal{C}} U \right)$$

Proof. Let $\varepsilon_0 > 0$ be small enough so that for all $0 < y \leq \varepsilon_0$ we have

$$\left(1 - \frac{y}{2q}\right)^q > 1 - y.$$

Let $z = z(\varepsilon) \in (0, 1)$ be small enough so that for a given $0 < \varepsilon \leq \varepsilon_0$ we have

$$\frac{1-z}{1+z}(1 - \varepsilon/2) > (1 - \varepsilon).$$

Lemma 2.5 tells us that there exists large enough $s = s(\varepsilon, X_0)$ so that

$$1 - z < \frac{\mu(B(x, t))}{c_X t^q} < 1 + z \quad \text{for all } x \in X_0, t \geq s/2,$$

where the constant $c_X \in \mathbb{R}_{>0}$ is given by the definition of metric measure space of exact polynomial volume growth. Then, for all $\delta \in (0, \min(\frac{\varepsilon}{4q}, \frac{1}{2})]$, $x \in X_0$ and $t > s$ we have

$$\frac{1-z}{1+z}(1 - \delta)^q < \frac{\mu(B(x, (1 - \delta)t))}{\mu(B(x, t))}.$$

Since

$$\frac{1-z}{1+z}(1 - \delta)^q > 1 - \varepsilon$$

for all $0 < \delta \leq \min(\frac{\varepsilon}{4q}, \frac{1}{2})$, this completes the proof. \square

As for δ -expansions, we state the following lemma.

Lemma 2.7 ('Controlling the size of δ -expansion'). *Let (X, μ) be a metric measure space of exact polynomial growth of degree $q > 0$ and let $X_0 \subseteq X$ be a bounded set. There exists a number $\varepsilon_0 \in (0, 1)$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there exist a number $s = s(\varepsilon, X_0)$ such that the following assertion holds. If \mathcal{C} is a countable collection of disjoint balls in X such that*

$$\text{rad}(\mathcal{C}) \geq s$$

and

$$\text{core}(\mathcal{C}) \subseteq X_0,$$

then for all $\delta \in (0, \frac{\varepsilon}{4q})$ we have

$$(2.4) \quad \mu \left(\bigsqcup_{U \in \mathcal{C}} U \right) \geq (1 - \varepsilon) \mu \left(\bigcup_{U \in \mathcal{C}} U^\delta \right).$$

The proof is almost identical to the proof of Lemma 2.6, so we leave it to the reader. The following lemma, whose proof is also straightforward, shows that for metric measure spaces of positive exact polynomial growth there *exist* sufficiently distant points in balls $B(x, r)$ as r is large enough.

Lemma 2.8. *Let (X, μ) be a metric measure space of exact polynomial growth of degree $q > 0$ and let $X_0 \subseteq X$ be a bounded set. There exists a number $s_* = s_*(X_0)$ such that the following assertion holds. If $B(x, r)$ is a ball of radius $r \geq s_*$ with center $x \in X_0$, then there exist points $y_1, y_2 \in B(x, r)$ such that $d(y_1, y_2) > \frac{2r}{3}$.*

Finally, we state the last lemma of the section. The proof is analogous to the proofs of the previous results, and we leave it to the reader as well.

Lemma 2.9. *Let (X, μ) be a metric measure space of exact polynomial growth of degree $q > 0$ and let $X_0 \subseteq X$ be a bounded set. There exists a number $s_! = s_!(X_0)$ such that the following assertion holds. If $B(x, r_1), B(x, r_2)$ are two balls with $x \in X_0$ such that $r_1, r_2 \geq s_!$, then*

$$\frac{\mu(B(x, r_1))}{\mu(B(x, r_2))} > 1 + t \quad \text{for some } t > 0$$

implies that

$$\frac{r_1}{r_2} > \left(\frac{1 + t}{1 + 2t/3} \right)^{1/q}.$$

Remark 2.10. Suppose that (X, μ) is a metric measure space of *uniformly* exact polynomial volume growth of degree $q > 0$. Then it is clear that one can remove the assumption that

$$\text{core}(\mathcal{C}) \subseteq X_0$$

for a bounded set X_0 in Lemmas 2.7 and 2.6. Furthermore, the numbers s depend on ε and space (X, μ) only. Similarly, the assumption that

$$x \in X_0$$

for a bounded set X_0 in Lemmas 2.8, 2.9 can be removed and the numbers $s_*, s_!$ depend on the space (X, μ) only.

2.2. Groups of Polynomial Growth. Let Γ be a finitely generated discrete group and $\{\gamma_1, \dots, \gamma_k\}$ be a fixed generating set. Each element $\gamma \in \Gamma$ can be written as a product $\gamma_{i_1}^{p_1} \gamma_{i_2}^{p_2} \dots \gamma_{i_l}^{p_l}$ for some indexes $i_1, i_2, \dots, i_l \in 1, \dots, k$ and some integers $p_1, p_2, \dots, p_l \in \mathbb{Z}$. We define the **norm** of an element $\gamma \in \Gamma$ by

$$\|\gamma\| := \inf \left\{ \sum_{i=1}^l |p_i| : \gamma = \gamma_{i_1}^{p_1} \gamma_{i_2}^{p_2} \dots \gamma_{i_l}^{p_l} \right\},$$

where the infimum is taken over all representations of γ as a product of the generating elements. The norm $\|\cdot\|$ on Γ can, in general, depend on the generating set, but it is easy to show [CSC10, Corollary 6.4.2] that two different generating sets produce equivalent norms. We will always say what generating set is used in the definition of a norm, but we will omit an explicit reference to the generating set later on.

The norm $\|\cdot\|$ yields a right invariant metric on Γ defined by

$$d_R(x, y) := \|xy^{-1}\| \quad \text{for } x, y \in \Gamma,$$

and a left invariant metric on Γ defined by

$$d_L(x, y) := \|x^{-1}y\| \quad \text{for } x, y \in \Gamma,$$

which we call the **word metrics**. The right invariance of d_R means that the right multiplication

$$R_g : \Gamma \rightarrow \Gamma, \quad x \mapsto xg \quad (x \in \Gamma)$$

is an isometry for every $g \in \Gamma$ with respect to d_R . Similarly, the left invariance of d_L means that the left multiplications are isometries with respect to d_L . We let $d := d_R$ and view Γ as a metric space with the metric d . For $x \in \Gamma$, $r \in \mathbb{R}_{\geq 0}$ let

$$B(x, r) := \{y \in \Gamma : d(x, y) \leq r\}$$

be the closed ball of radius r with center x with respect to d . Let $e \in \Gamma$ be the neutral element. It is clear that

$$B(n) = \{y : d_R(e, y) \leq n\} = \{y : d_L(e, y) \leq n\} = \{y : \|y\| \leq n\},$$

i.e., the ball $B(n)$ is precisely the ball $B(e, n)$ with respect to the left and the right word metric. Furthermore,

$$|B(x, n)| = |B(n)|$$

for all $x \in \Gamma$, $n \geq 0$.

We say that the group Γ is of **polynomial growth** if there are constants $C, d > 0$ such that for all $n \geq 1$ we have

$$|B(n)| \leq Cn^d.$$

Example 2.11. Consider the group \mathbb{Z}^d for $d \in \mathbb{N}$ and let $\gamma_1, \dots, \gamma_d \in \mathbb{Z}^d$ be the standard basis elements of \mathbb{Z}^d . That is, γ_i is defined by

$$\gamma_i(j) := \delta_i^j \quad (j = 1, \dots, d)$$

for all $i = 1, \dots, d$. We consider the generating set given by elements $\sum_{k \in I} (-1)^{\varepsilon_k} \gamma_k$

for all subsets $I \subseteq [1, d]$ and all functions $\varepsilon : [1, d] \rightarrow \{0, 1\}$. Then it is easy to see by induction on dimension that $B(n) = [-n, \dots, n]^d$, hence

$$|B(n)| = (2n + 1)^d \quad \text{for all } n \in \mathbb{N}$$

with respect to this generating set, i.e., \mathbb{Z}^d is a group of polynomial growth.

Let $d \in \mathbb{Z}_{\geq 0}$. We say that the group Γ has **polynomial growth of degree d** if there is a constant $C > 0$ such that

$$\frac{1}{C}n^d \leq |B(n)| \leq Cn^d \quad \text{for all } n \in \mathbb{N}.$$

It was shown in [Bas72] that, if Γ is a finitely generated nilpotent group, then Γ has polynomial growth of some degree $d \in \mathbb{Z}_{\geq 0}$. Furthermore, one can show [CSC10, Proposition 6.6.6] that if Γ is a group and $\Gamma' \leq \Gamma$ is a finite index, finitely generated nilpotent subgroup, having polynomial growth of degree $d \in \mathbb{Z}_{\geq 0}$, then the group Γ has polynomial growth of degree d . The converse is true as well: it was proved in [Gro81] that, if Γ is a group of polynomial growth, then there exists a finite index, finitely generated nilpotent subgroup $\Gamma' \leq \Gamma$. It follows that if Γ is a group of polynomial growth with the growth function γ , then there is a constant $C > 0$ and an integer $d \in \mathbb{Z}_{\geq 0}$, called the **degree of polynomial growth**, such that

$$\frac{1}{C}n^d \leq |B(n)| \leq Cn^d \quad \text{for all } n \in \mathbb{N}.$$

An even stronger result was obtained in [Pan83], where it is shown that, if Γ is a group of polynomial growth of degree $d \in \mathbb{Z}_{\geq 0}$, then the limit

$$(2.5) \quad c_\Gamma := \lim_{n \rightarrow \infty} \frac{|B(n)|}{n^d}$$

exists. It follows that a group of polynomial growth of degree $d \in \mathbb{Z}_{\geq 0}$ is a metric measure space of uniformly exact polynomial volume growth of degree d .

3. COVERING LEMMAS

First, we state the basic finitary Vitali covering lemma. The proof is well-known, so we omit it.

Lemma 3.1. *Let $\mathcal{C} := \{B_1, \dots, B_n\}$ be a finite collection of balls in a metric space X . Then there exists a disjoint subcollection $\mathcal{C}_0 \subseteq \mathcal{C}$ such that*

$$\bigcup_{W \in \mathcal{C}} W \subseteq \bigcup_{W \in \mathcal{C}_0} 3 \cdot W.$$

As a consequence, we derive the following version of the Vitali lemma, which we will later use in the proofs.

Lemma 3.2. *Let (X, μ) be a metric measure space of exact polynomial growth of degree q and let $X_0 \subseteq X$ be a bounded set. Then there exists a number $s' = s'(X_0) > 0$ such that the following assertion holds. If \mathcal{C} is a finite collection of balls in X such that*

$$\text{rad}(\mathcal{C}) \geq s'$$

and

$$\text{core}(\mathcal{C}) \subseteq X_0,$$

then there exists a disjoint subcollection $\mathcal{C}_0 \subseteq \mathcal{C}$ such that

$$\mu\left(\bigcup_{V \in \mathcal{C}_0} V\right) \geq \frac{1}{3^{q+1}} \mu\left(\bigcup_{V \in \mathcal{C}} V\right)$$

Proof. Let $s' \in \mathbb{R}_{>0}$ be large enough such that for all balls $V = B(x, s)$ with $x \in X_0$ and $s \geq s'$ we have

$$(3.1) \quad \mu(3 \cdot V) < 3^{q+1} \mu(V).$$

Such s' exists because the volume growth is uniformly polynomially exact on X_0 . Next, given a collection \mathcal{C} satisfying the assertions of the theorem, we use Lemma 3.1 to obtain a disjoint subcollection $\mathcal{C}_0 \subseteq \mathcal{C}$ such that

$$\bigcup_{V \in \mathcal{C}} V \subseteq \bigcup_{V \in \mathcal{C}_0} 3 \cdot V.$$

Combining this with Equation (3.1), we obtain the statement of the lemma. \square

Remark 3.3. Similar to Remark 2.10 above we can note that for metric measure spaces of *uniformly* exact polynomial growth the assumption in Lemma 3.2 that

$$\text{core}(\mathcal{C}) \subseteq X_0$$

for a bounded set X_0 can be dropped. In this case the number s' does depends only on the space (X, μ) .

When the original collection \mathcal{C} in Lemma 3.2 satisfies certain additional assumptions, applying the lemma multiple times can yield a disjoint subcollection which ‘almost covers’ $\bigcup_{V \in \mathcal{C}} V$. In order to make this precise, we start with some definitions.

Let X be a metric space, $W \subseteq X$ be a nonempty subset and $n \in \mathbb{N}$ be an integer. A **tower** \mathcal{U} over W is a finite collection of balls

$$\mathcal{U} = \{U_i(x) : x \in W, i = 1, \dots, n\}$$

indexed by points $x \in W$ and integers $i = 1, \dots, n$, which satisfies the following assertions:

- 1) $x \in U_i(x)$ for all $x \in W$ and all $i = 1, \dots, n$;
- 2) $U_1(x) \subseteq U_2(x) \subseteq \dots \subseteq U_n(x)$ for all $x \in W$.

The number n is called the **height** of a tower, and we denote the height of a tower \mathcal{U} by $\text{ht}(\mathcal{U})$. We stress that the set W might be infinite, but a tower \mathcal{U} over W is always a finite collection of balls. A tower \mathcal{U} over W of height n is called **centered** if for every point $x \in W$ and every index $i = 1, \dots, n$ the ball $U_i(x)$ is of the form $B(x, r)$ for some $r \geq 0$. For a tower \mathcal{U} of height n and every index $i = 1, \dots, n$ let

$$\mathcal{U}_i := \{U_i(x) : U_i(x) \in \mathcal{U}\}$$

be the i -th level of the tower \mathcal{U} and let

$$\mathcal{U}_{\leq i} := \bigcup_{j=1}^i \mathcal{U}_j,$$

$$\mathcal{U}_{\geq i} := \bigcup_{j=i}^n \mathcal{U}_j$$

be the towers obtained from the tower \mathcal{U} by taking the first i levels and the last $n - i + 1$ levels respectively. The balls in \mathcal{U}_i will be called **i -th level balls**. Given a number $\delta \in \mathbb{R}_{\geq 0}$, a tower \mathcal{U} over $W \subseteq X$ of height $n \in \mathbb{N}$ is called **δ -expanding** if

$$(3.2) \quad U_i(x)^\delta \subseteq U_{i+1}(x) \quad \text{for all } x \in W, i = 1, \dots, n-1.$$

Finally, given a collection of balls \mathcal{C} , we denote by $[\mathcal{C}]$ the collection of all maximal balls in \mathcal{C} with respect to the set inclusion, i.e.,

$$[\mathcal{C}] = \{U \in \mathcal{C} : U \subseteq V \Rightarrow U = V \text{ for all } V \in \mathcal{C}\}.$$

We will call the balls in $[\mathcal{C}]$ **maximal**. It is clear that if \mathcal{U} is a tower, then

$$\bigcup_{U \in [\mathcal{U}_n]} U = \bigcup_{U \in \mathcal{U}} U.$$

Lemma 3.4. *Let X be a metric space and $\varepsilon \in (0, 1)$ be arbitrary. Suppose that \mathcal{U} is a $(1 + \frac{4}{\varepsilon})$ -expanding tower over some subset $W \subseteq X$ of height $n > 1$. Then for all indexes $1 \leq i < j \leq n$ and for all balls $U \in \mathcal{U}_i$, $V \in [\mathcal{U}_j]$ we have*

$$U \cap V \neq \emptyset \Rightarrow U \subseteq V^\varepsilon.$$

Proof. Suppose that $U = B(x, s) \in \mathcal{U}_i$ and $V = B(y, t) \in [\mathcal{U}_j]$ are some balls such that

$$U \cap V \neq \emptyset$$

and U is *not* a subset of V^ε . Then the ball U does not fit into the annulus $V^\varepsilon \setminus V$ of width εt , hence

$$\varepsilon t \leq 2s.$$

This implies that $s \geq \frac{\varepsilon t}{2}$, hence $(2 + \frac{4}{\varepsilon})s \geq 2s + 2t$. We conclude that $V \subsetneq U^{1+\frac{4}{\varepsilon}}$, which is a contradiction since the ball V is maximal and the ball $U^{1+\frac{4}{\varepsilon}}$ is contained in some ball from \mathcal{U}_j by the definition of a tower. \square

To simplify the presentation, we fix the following notation till the end of this section. Let (X, μ) be an arbitrary metric measure space of exact polynomial growth of degree $q \in \mathbb{R}_{>0}$ and let $X_0 \subseteq X$ be a bounded set. Define the constant

$$C := \frac{3^{q+1}}{3^{q+1} - 1}.$$

After these preparations, we can prove the so-called effective Vitali covering theorem. The proof is based on the proof of the effective Vitali covering lemma from [Mor16], where the results of S. Kalikow and B. Weiss from [KW99] were generalized.

Theorem 3.5 (‘Effective Vitali Covering’). *Let $\varepsilon \in (0, 1)$ be small enough. Then there exist a number $s_0 = s_0(\varepsilon, X_0)$ such that for a tower \mathcal{U} satisfying the assertions*

- 1) \mathcal{U} is $(1 + \frac{36q}{\varepsilon})$ -expanding;
- 2) $\text{core}(\mathcal{U}) \subseteq X_0$;
- 3) $\text{rad}(\mathcal{U}) \geq s_0$;
- 4) $\text{ht}(\mathcal{U}) \geq 1 + \log_C \frac{2}{\varepsilon}$

there exists a disjoint subcollection $\mathcal{V} \subseteq \mathcal{U}$ such that

$$\mu \left(\bigcup_{V \in \mathcal{V}} V \right) \geq (1 - \varepsilon) \mu \left(\bigcup_{W \in \mathcal{U}_1} W \right)$$

Proof. First of all, we specify some of the parameters of the lemma. By ‘small enough ε ’ we mean that $\varepsilon < \varepsilon_0$, where ε_0 is given by Lemma 2.7. Let s' be the parameter provided by Lemma 3.2 and $s = s(\varepsilon/2, X_0)$ be the parameter provided by Lemma 2.7. We define

$$s_0(\varepsilon, X_0) := \max(s', s(\varepsilon/2, X_0)).$$

Let $n := \text{ht}(\mathcal{U})$ be height of the tower \mathcal{U} and let $\delta := \frac{\varepsilon}{9q}$. For each $i = 1, \dots, n$ let

$$U_i := \bigcup_{V \in \mathcal{U}_i} V$$

be the union of i -th level balls. The goal is to show that there exists a collection of disjoint balls $\mathcal{V} \subseteq \mathcal{U}$ such that

$$(3.3) \quad \mu \left(\bigcup_{V \in \mathcal{V}} V^\delta \right) \geq (1 - \varepsilon/2) \mu \left(\bigcup_{W \in \mathcal{U}_1} W \right).$$

Once such a collection is found, an application of Lemma 2.7 would complete the proof of the theorem since $\delta < \frac{1}{4q} \cdot \frac{\varepsilon}{2}$.

The main idea of the proof is to cover a positive fraction of U_n by disjoint union of maximal n -level balls from \mathcal{U}_n using Lemma 3.2, then cover a positive fraction of the remainder of U_{n-1} by a disjoint union of $(n-1)$ -level balls and so on. We make this intuition formal below.

Let $\mathcal{C}_n \subseteq \mathcal{U}_n$ be the collection of disjoint balls, which we obtain by applying Lemma 3.2 to the collection of maximal n -level balls $[\mathcal{U}_n]$. It follows that

$$\mu \left(\bigcup_{V \in \mathcal{C}_n} V \right) \geq \frac{1}{3^{q+1}} \mu(U_n)$$

Let $S_n := \bigcup_{V \in \mathcal{C}_n} V$. The computation above shows that

$$(3.4) \quad S_n \text{ covers at least a } \frac{1}{3^{q+1}}\text{-fraction of } U_n$$

and

$$(3.5) \quad \mu(U_1) - \mu(S_n) \leq \mu(U_1) - \frac{1}{3^{q+1}} \mu(U_1) = C^{-1} \mu(U_1).$$

In general, suppose by induction that the collections of disjoint balls

$$\mathcal{C}_n, \mathcal{C}_{n-1}, \dots, \mathcal{C}_{n-k+1}$$

with the respective unions

$$S_n, S_{n-1}, \dots, S_{n-k+1}$$

have been constructed, where $k \geq 1$. Define the union of the corresponding δ -expansions

$$I_{k-1} := \bigcup_{V \in \mathcal{C}_n \cup \dots \cup \mathcal{C}_{n-k+1}} V^\delta.$$

By induction, we assume that

$$(3.6) \quad \mu(I_{k-1}) \geq (1 - C^{-k+1})\mu(U_1).$$

Let

$$\begin{aligned} \tilde{\mathcal{C}}_{n-k} &:= \{V : V \in [\mathcal{U}_{n-k}] \text{ is a maximal } (n-k) \text{--level ball} \\ &\quad \text{such that } V \cap (S_n \cup S_{n-1} \cup \dots \cup S_{n-k+1}) = \emptyset\} \end{aligned}$$

be the collection of all maximal $(n-k)$ -level balls disjoint from $S_n \cup S_{n-1} \cup \dots \cup S_{n-k+1}$ and let \tilde{S}_{n-k} be its union. We apply Lemma 3.2 once to obtain a collection $\mathcal{C}_{n-k} \subseteq \tilde{\mathcal{C}}_{n-k}$ of pairwise disjoint balls such that

$$\mu\left(\bigsqcup_{V \in \mathcal{C}_{n-k}} V\right) \geq \frac{1}{3^{q+1}}\mu(\tilde{S}_{n-k})$$

and let $S_{n-k} := \bigsqcup_{V \in \mathcal{C}_{n-k}} V$. We want to show that

$$(3.7) \quad \mu\left(\bigcup_{V \in \mathcal{C}_n \cup \dots \cup \mathcal{C}_{n-k+1}} V^\delta \cup S_{n-k}\right) \geq (1 - C^{-k})\mu(U_1)$$

and to do so it suffices to prove that

$$(3.8) \quad \mu\left(\bigcup_{V \in \mathcal{C}_n \cup \dots \cup \mathcal{C}_{n-k+1}} V^\delta \cup S_{n-k}\right) \geq \mu(I_{k-1}) + \frac{1}{3^{q+1}}\mu(U_{n-k} \setminus I_{k-1}),$$

due to the inductive assertion above (Equation (3.6)). We decompose the set $U_{n-k} \setminus I_{k-1}$ as follows:

$$U_{n-k} \setminus I_{k-1} = \tilde{S}_{n-k} \sqcup \left(U_{n-k} \setminus (I_{k-1} \cup \tilde{S}_{n-k})\right).$$

The part $U_{n-k} \setminus (I_{k-1} \cup \tilde{S}_{n-k})$ is covered by the $(n-k)$ -level balls intersecting $S_n \cup S_{n-1} \cup \dots \cup S_{n-k+1}$. Hence, due to Lemma 3.4, the set $U_{n-k} \setminus (I_{k-1} \cup \tilde{S}_{n-k})$ is covered by the δ -expansions of balls in $\mathcal{C}_n \cup \dots \cup \mathcal{C}_{n-k+1}$. Next, S_{n-k} covers at least a $\frac{1}{3^{q+1}}$ -fraction of the set \tilde{S}_{n-k} . It follows that the set

$$\bigcup_{V \in \mathcal{C}_n \cup \dots \cup \mathcal{C}_{n-k+1}} V^\delta \cup S_{n-k}$$

covers the set I_{k-1} and at least a $\frac{1}{3^{q+1}}$ -fraction of the set $U_{n-k} \setminus I_{k-1}$. Thus we have proved inequalities (3.8) and (3.7).

It is obvious that one can continue in this way down to the 1-st level balls. This would yield a collection of maximal balls

$$\mathcal{C} := \bigcup_{i=1}^n \mathcal{C}_i$$

so that the measure of the union of δ -expansions of balls in \mathcal{C} is at least $(1 - C^{-n})$ times the measure of U_1 . Since $n \geq 1 + \log_C \frac{2}{\varepsilon}$, we deduce that the proof of the inequality (3.3) is complete. \square

Remark 3.6. We note that for metric measure spaces of *uniformly* exact polynomial growth the assumption in Theorem 3.5 that

$$\text{core}(\mathcal{U}) \subseteq X_0$$

for a bounded set X_0 can be dropped. In this case the parameter s_0 depends on ε and the space (X, μ) only.

Theorem 3.5 will play an essential role in the proof of the following result. For convenience, we fix some additional notation till the end of the section. We let $0 < \varepsilon < \varepsilon_0$ be arbitrary, and define

$$s_1 := \max(s_0(\varepsilon, X_0), s_*(X_0))$$

to be the maximum of the corresponding parameters provided by Theorem 3.5 and Lemma 2.8. Fix $L \in \mathbb{Z}_{\geq 0}$.

Assume that \mathcal{U} and \mathcal{V} are two centered towers over the set $X_0 \subseteq X$ of height $L + 1$ such that

$$\text{core}(\mathcal{U}), \text{core}(\mathcal{V}) \subseteq X_0,$$

$$\text{rad}(\mathcal{U}), \text{rad}(\mathcal{V}) \geq s_1$$

and

$$U_1(x) \subseteq V_1(x) \subseteq U_2(x) \subseteq V_2(x) \subseteq \cdots \subseteq U_{L+1}(x) \subseteq V_{L+1}(x)$$

for all points $x \in X_0$. Let

$$\Delta := \max(144q, 4).$$

Suppose that

$$(3.9) \quad U_i^{(1+\Delta/\varepsilon)}(x) \subseteq V_i(x)$$

for all indexes $i = 1, \dots, L + 1$ and all points $x \in X_0$. It is clear that, since the towers are centered, this implies that the towers \mathcal{U}, \mathcal{V} are $(1 + \frac{\Delta}{\varepsilon})$ -expanding.

Lemma 3.7. *Suppose that $L \geq 1 + 2 \log_C \frac{2}{\varepsilon}$. Then either there exists a ball $W \in \mathcal{V}_{L+1}$ which can be 4ε -filled by a disjoint subcollection of \mathcal{U} , or, otherwise, the inequality*

$$(3.10) \quad \mu \left(\bigcup_{V \in \mathcal{V}} V \right) \geq \left(1 + \frac{\varepsilon}{3^{q+1}} \right) \mu \left(\bigcup_{U \in \mathcal{U}_1} U \right)$$

holds.

Proof. First of all, we apply Lemma 3.2 to the collection $[\mathcal{V}_{L+1}]$, obtaining a collection of maximal disjoint balls $\mathcal{W} \subseteq [\mathcal{V}_{L+1}]$ such that

$$\mu \left(\bigcup_{W \in \mathcal{W}} W \right) \geq \frac{1}{3^{q+1}} \mu \left(\bigcup_{V \in [\mathcal{V}]} V \right) \geq \frac{1}{3^{q+1}} \mu \left(\bigcup_{U \in \mathcal{U}_1} U \right).$$

There are two further possibilities. First, suppose that for every $W \in \mathcal{W}$ we have

$$\mu \left(W \setminus \bigcup_{U \in \mathcal{U}_{\lfloor L/2 \rfloor}} U \right) \geq \varepsilon \mu(W).$$

Let $D := \bigcup_{U \in \mathcal{U}_{\lfloor L/2 \rfloor}} U$. Then $\bigcup_{U \in \mathcal{U}_1} U \subseteq D$, and so

$$\begin{aligned} \mu \left(\bigcup_{V \in \mathcal{V}} V \right) &\geq \mu \left(\bigcup_{U \in \mathcal{U}_1} U \right) + \mu \left(\bigcup_{W \in \mathcal{W}} W \setminus D \right) = \\ &= \mu \left(\bigcup_{U \in \mathcal{U}_1} U \right) + \sum_{W \in \mathcal{W}} \mu(W \setminus D) \geq \\ &\geq \left(1 + \frac{\varepsilon}{3^{q+1}} \right) \mu \left(\bigcup_{U \in \mathcal{U}_1} U \right). \end{aligned}$$

This is precisely inequality (3.10) in the statement of the lemma. Otherwise, suppose that the ball $W \in \mathcal{W}$ is such that

$$\mu \left(W \setminus \bigcup_{U \in \mathcal{U}_{\lfloor L/2 \rfloor}} U \right) < \varepsilon \mu(W).$$

We intend to prove that W can be 4ε -filled by a disjoint subcollection of \mathcal{U} . Define the collections of balls

$$\mathcal{Y} := \{U_i(x) : 1 \leq i \leq \lfloor L/2 \rfloor, x \in X_0 \text{ are such that } U_{\lfloor L/2 \rfloor}(x) \cap W \neq \emptyset\}$$

and

$$\mathcal{Z} := \{U_i(x) : 1 \leq i \leq L, x \in X_0 \text{ are such that } U_L(x) \subseteq W\}.$$

Observe that \mathcal{Z} is in fact a *tower*. Furthermore,

$$(3.11) \quad \mu \left(W \cap \bigcup_{Y \in \mathcal{Y}} Y \right) \geq (1 - \varepsilon) \mu(W).$$

We will show that

$$(3.12) \quad \mu \left(W \cap \bigcup_{Z \in \mathcal{Z}_{\leq \lfloor L/2 \rfloor}} Z \right) \geq (1 - 3\varepsilon) \mu(W).$$

Assume that this does not hold. Lemma 2.6 implies that

$$\mu(\text{int}_{\varepsilon/4q}(W)) \geq (1 - \varepsilon) \mu(W).$$

Then there exists a subset $W_0 \subseteq \text{int}_{\varepsilon/4q}(W)$ such that

$$W_0 \cap \bigcup_{Z \in \mathcal{Z}_{\leq \lfloor L/2 \rfloor}} Z = \emptyset$$

and $\mu(W_0) \geq 2\varepsilon \mu(W)$. Condition (3.11) implies that there exists a ball $U_i(x) \in \mathcal{Y} \setminus \mathcal{Z}_{\leq \lfloor L/2 \rfloor}$ such that $U_i(x) \cap W_0 \neq \emptyset$. Let $z_1 \in U_i(x) \cap W_0$. Since $U_i(x) \notin \mathcal{Z}_{\leq \lfloor L/2 \rfloor}$, we know that $U_L(x) \cap W^c \neq \emptyset$. Let $z_2 \in U_L(x) \cap W^c$, then, clearly, $z_1, z_2 \in U_L(x)$.

Suppose for the moment that $W = B(y, r)$ for some $y \in X_0$ and some $r > 0$. Since $W_0 \subseteq \text{int}_{\varepsilon/4q}(W)$, we have

$$d(z_1, z_2) \geq \frac{\varepsilon r}{4q},$$

and hence the radius of the ball $U_L(x)$ is greater or equal than $\frac{\varepsilon r}{8q}$. If $V_L(x) = B(x, r')$ for some $r' > 0$, then, since $U_L^{(1+144q/\varepsilon)}(x) \subseteq V_L(x)$, we conclude using Lemma 2.8 that

$$r' \geq \left(2 + \frac{144q}{\varepsilon}\right) \cdot \frac{1}{3} \cdot \frac{\varepsilon r}{8q} \geq 6r.$$

The intersection $V_L(x) \cap W$ is nonempty, since it contains point z_1 . The tower \mathcal{V} is $(1 + \frac{4}{\varepsilon})$ -expanding as well, hence $V_L(x) \subseteq W^\varepsilon$ due to Lemma 3.4. This contradicts to the estimate $r' \geq 6r$ above. Thus we have proved the estimate (3.12).

Now consider the tower $\tilde{\mathcal{Z}} := \mathcal{Z}_{\geq \lfloor L/2 \rfloor}$, which has the height of at least $1 + \log_C \frac{2}{\varepsilon}$. Due to the estimate (3.12), the first level balls of the tower $\tilde{\mathcal{Z}}$ cover at least a $(1 - 3\varepsilon)$ -fraction of W . We apply Theorem 3.5 to the tower $\tilde{\mathcal{Z}}$ and obtain a disjoint collection of balls \mathcal{Z}' , which are all contained in W , such that

$$\mu \left(\bigcup_{Z \in \mathcal{Z}'} Z \right) \geq (1 - \varepsilon) \mu \left(\bigcup_{Z \in \tilde{\mathcal{Z}}_1} Z \right) \geq (1 - 4\varepsilon) \mu(W).$$

This shows that the ball W can be 4ε -filled by a disjoint collection of balls from \mathcal{U} , and the proof is complete. \square

For the convenience of the reader we recollect all assumptions in the statement of the following main theorem of the section. The proof is a minor adaptation of the one in [Hoc09]. We remind the reader that the constants C, Δ are defined as

$$C := \frac{3^{q+1}}{3^{q+1} - 1}, \quad \Delta := \max(144q, 4),$$

and we define a new constant $K := \lceil 2 \log_C \frac{4}{\varepsilon} \rceil$.

Theorem 3.8. *Let (X, μ) be a metric measure space of exact polynomial growth of degree $q \in \mathbb{R}_{>0}$ and let $X_0 \subseteq X$ be a bounded set. Then for all $\varepsilon > 0$ small enough there exists $s_1 := s_1(X_0, \varepsilon)$ such that the following holds. Let $L \geq 2 \log_C(4/\varepsilon)$ be an arbitrary integer. Assume that \mathcal{U} and \mathcal{V} are centered towers over X_0 of height $L + 1$ such that*

$$\begin{aligned} \text{core}(\mathcal{U}), \text{core}(\mathcal{V}) &\subseteq X_0, \\ \text{rad}(\mathcal{U}), \text{rad}(\mathcal{V}) &\geq s_1 \end{aligned}$$

and

$$U_1(x) \subseteq V_1(x) \subseteq U_2(x) \subseteq V_2(x) \subseteq \cdots \subseteq U_{L+1}(x) \subseteq V_{L+1}(x) \quad \text{for all } x \in X_0.$$

Suppose that

$$(3.13) \quad U_i^{(1+\Delta/\varepsilon)}(x) \subseteq V_i(x) \quad \text{for all } x \in X_0, i = 1, \dots, L + 1$$

and that no ball $V \in \mathcal{V}$ can be 4ε -filled by a disjoint subcollection of \mathcal{U} . Then

$$\mu \left(\bigcup_{U \in \mathcal{U}_1} U \right) \leq \left(1 + \frac{\varepsilon}{3^{q+1}} \right)^{-\lfloor L/K \rfloor} \mu \left(\bigcup_{V \in \mathcal{V}_{L+1}} V \right).$$

Proof. We assume that $0 < \varepsilon < \varepsilon_0$ (with ε_0 determined in Theorem 3.5). Consider the case $L := K$ first. Then the statement of the theorem is precisely Lemma 3.7, since $\lceil 2 \log_C \frac{4}{\varepsilon} \rceil \geq 1 + 2 \log_C \frac{2}{\varepsilon}$. In general, we suppose that $iK \leq L < (i+1)K$ for some uniquely determined $i \in \mathbb{N}$ and proceed by induction on i . So suppose that we have proved the statement for $i \geq 1$ and want to proceed to $i+1$. Let \mathcal{U}, \mathcal{V} be two arbitrary towers of height n such that $1 + (i+1)K \leq n < 1 + (i+2)K$, satisfying the assertions of the theorem. Then

$$\tilde{\mathcal{U}} := \mathcal{U}_{\leq n-K}, \quad \tilde{\mathcal{V}} := \mathcal{V}_{\leq n-K}$$

are two towers of height $n - K$, satisfying the assertions of the theorem. The inductive assumption implies that

$$\mu \left(\bigcup_{U \in \tilde{\mathcal{U}}_1} U \right) \leq \left(1 + \frac{\varepsilon}{3^{q+1}} \right)^{-i} \mu \left(\bigcup_{V \in \tilde{\mathcal{V}}_{n-K}} V \right) \leq \left(1 + \frac{\varepsilon}{3^{q+1}} \right)^{-i} \mu \left(\bigcup_{U \in \mathcal{U}_{n-K+1}} U \right).$$

Now observe that the towers $\mathcal{V}' := \mathcal{V}_{\geq n-K+1}, \mathcal{U}' := \mathcal{U}_{\geq n-K+1}$ of height K satisfy the assertions of the theorem as well. Thus we can apply the base case of the induction, and so

$$\mu \left(\bigcup_{U \in \mathcal{U}_{n-K+1}} U \right) \leq \left(1 + \frac{\varepsilon}{3^{q+1}} \right)^{-1} \mu \left(\bigcup_{V \in \mathcal{V}_n} V \right).$$

Combining this with the previous inequality, we deduce that

$$\mu \left(\bigcup_{U \in \mathcal{U}_1} U \right) \leq \left(1 + \frac{\varepsilon}{3^{q+1}} \right)^{-i-1} \mu \left(\bigcup_{V \in \mathcal{V}_n} V \right),$$

and the proof is complete. \square

Remark 3.9. Similar to the previous remarks, we observe that for metric measure spaces of *uniformly* exact polynomial growth the assumption in Theorem 3.8 that

$$\text{core}(\mathcal{U}), \text{core}(\mathcal{V}) \subseteq X_0$$

for a bounded set X_0 can be dropped. In this case the parameter s_1 depends on ε and the space (X, μ) only.

4. HOCHMAN'S AND KINGMAN'S THEOREMS

4.1. Hochman's Upcrossing Theorem. The goal of this section is to prove the following theorem, which we stated in the introduction.

Theorem 4.1. *Let Γ be a group of polynomial growth. For all intervals $(\alpha, \beta) \subset \mathbb{R}$ and all $\delta > 0$ small enough there exist constants $c > 0$ and $\rho \in (0, 1)$ such that the following holds. If $(S_{B(g,r)})$ is a stationary process, then for all $k \geq 1$*

$$\begin{aligned} & \mathbb{P}(\{x : (S_{B(i)}(x))_{i \geq 1} \in \mathcal{F}_{(\alpha, \beta)}^k\}) \leq \\ & \leq c\rho^k + \mathbb{P}\left(\left\{x : \begin{array}{l} \exists n > k \text{ s.t. } S_{B(n)}(x) > \beta \text{ and } B(n) \text{ can be } \delta\text{-filled} \\ \text{by disjoint balls } V_1, \dots, V_m \text{ s.t. } \forall i \ S_{V_i}(x) < \alpha \end{array} \right\}\right). \end{aligned}$$

We proceed to the proof. Without loss of generality we assume that Γ is group of polynomial growth of degree $q \geq 1$. For all $k, l \in \mathbb{N}$ and all $g \in \Gamma$ we define the events

$$Q_{g,l}^k := \{x : (S_{B(g,i)}(x))_{i=1}^l \in \mathcal{F}_{(\alpha, \beta)}^k\}$$

and

$$R_g^k := \left\{x : \begin{array}{l} \exists n > k \text{ s.t. } S_{B(g,n)}(x) > \beta \text{ and } B(g, n) \text{ can be } \delta\text{-filled} \\ \text{by disjoint balls } V_1, \dots, V_m \text{ s.t. } \forall i \ S_{V_i}(x) < \alpha \end{array} \right\}.$$

Let $Q_g^k := \bigcup_{l \geq 1} Q_{g,l}^k$ be the increasing union of the events $Q_{g,l}^k$ for all $l \geq 1$. The goal is to show existence of universal constants $c > 0, \rho \in (0, 1)$ such that that

$$\mathbb{P}(Q_e^k) \leq c\rho^k + \mathbb{P}(R_e^k) \quad \text{for all } k \geq 1,$$

and it is clear that to do so it suffices to prove that for all l we have

$$\mathbb{P}(Q_{e,l}^k \setminus R_e^k) \leq c\rho^k \quad \text{for all } k \geq 1.$$

The main idea of the proof is to use a ‘transference principle’. For integers $l, M \in \mathbb{N}$ and a point $x \in X$ we define the set

$$E_{M,l,x}^k := \{g : x \in Q_{g,l}^k \setminus R_g^k \text{ and } \|g\| \leq M\} \subseteq B(M).$$

The lemma below tells us essentially that each universal upper bound on the density of $E_{M,l,x}^k$ in $B(M)$ bounds the probability of $Q_{e,l}^k \setminus R_e^k$ from above as well.

Lemma 4.2 (‘Transference principle’). *Suppose that for a given constant $t \in \mathbb{R}_{\geq 0}$ the following holds: there is some $M_0 \in \mathbb{N}$ such that for all $M \geq M_0, k \geq 1$ and for \mathbb{P} -almost all $x \in X$ we have*

$$(4.1) \quad |E_{M,l,x}^k| \leq t |B(M)| + o_{x,l}(|B(M)|),$$

where $o_{x,l}(|B(M)|)/|B(M)|$ converges to 0 uniformly in x . Then

$$\mu(Q_{e,l}^k \setminus R_e^k) \leq t$$

for all $k \geq 1$.

Proof. Indeed, since the process is stationary, we have

$$\sum_{g \in B(M)} \int_X \mathbf{1}_{Q_{g,l}^k \setminus R_g^k}(x) d\mu = |B(M)| \mu(Q_{e,l}^k \setminus R_e^k)$$

for all $M \geq 1$. Then

$$\begin{aligned} \mu(Q_{e,l}^k \setminus R_e^k) &= \int_X \left(\frac{1}{|B(M)|} \sum_{g \in B(M)} \mathbf{1}_{Q_{g,l}^k \setminus R_g^k}(x) \right) d\mu \leq \\ &\leq \int_X \left(\frac{|E_{M,l,x}|}{|B(L)|} \right) d\mu, \end{aligned}$$

and the proof is complete since M can be arbitrarily large. \square

The goal now is to derive estimate (4.1) such that the assertions of Lemma 4.2 hold, i.e., to prove that for some universal constants $c > 0, \rho \in (0, 1)$ we have that

$$|E_{M,l,x}^k| \leq c\rho^k |B(M)| + o_{x,l}(|B(M)|)$$

holds for all M large enough, all k and almost all x . Observe that it suffices to do so for all k ‘large enough’ and change the universal constant c if necessary. Fix an arbitrary $x \in X$.

For each $g \in E_{M,l,x}^k$ there exists a sequence of balls $U_1(g) \subsetneq V_1(g) \subsetneq \dots \subsetneq U_k(g) \subsetneq V_k(g)$ with center g such that for every

$$S_{U_i}(x) < \alpha \text{ and } S_{V_i(g)}(x) > \beta \quad \text{for all } i = 1, \dots, k.$$

It is clear that for every $g \in E_{M,l,x}^k$ and every index $i = 1, \dots, k-1$ we have

$$(4.2) \quad |V_{i+1}(g)| > |U_{i+1}(g)| > |V_i(g)| > |U_i(g)|.$$

Since Γ is a group of polynomial growth endowed with a word metric and the balls $U_{i+1}(g), U_i(g), V_{i+1}(g)$ and $V_i(g)$ above are centered at g , condition (4.2) implies that the radius of the ball $U_{i+1}(g)$ is greater or equal than the radius of the ball $U_i(g)$ plus 2. Similarly, the radius of $V_{i+1}(g)$ is at least the radius of $V_i(g)$ plus 2.

Let $s_1 := \max(s_1(\delta/4), s_1)$ be the maximum of the constants given by Theorem 3.8 and Lemma 2.9. We skip the first $k' := \max(\lceil s_1 \rceil, \lceil k/2 \rceil)$ upcrossings to ensure that the radius of the balls $U_i(g), V_i(g)$ is not smaller than s_1 and k . Observe that $\frac{|V_i(g)|}{|U_i(g)|} > 1 + \delta$ for all indexes i and all $g \in E_{M,l,x}^k$. Indeed, suppose on the contrary that $\frac{|V_i(g)|}{|U_i(g)|} \leq 1 + \delta$ for some index i and some $g \in E_{M,l,x}^k$. Then $U_i(g)$ δ -fills $V_i(g)$ and the radius of $V_i(g)$ is at least k , thus leading to a contradiction, since $x \notin R_g^k$ by definition of $E_{M,l,x}^k$. Lemma 2.9 now implies that if $V_i(g) = B(g, r_1)$ and $U_i(g) = B(g, r_2)$, then

$$\frac{r_1}{r_2} > \left(\frac{1 + \delta}{1 + 2\delta/3} \right)^{1/q}.$$

Let

$$D := q \left\lceil \frac{\log(2 + 576q/\delta)}{\log(1 + \delta) - \log(1 + 2\delta/3)} \right\rceil + 1 \in \mathbb{N}.$$

We define towers \tilde{U}, \tilde{V} over the set $E_{M,l,x}^k$ of height $L = \lfloor (k - k')/D \rfloor$ by setting

$$\tilde{U}_i(g) := U_{D(i-1)+1}(g), \quad \tilde{V}_i(g) := V_{D(i-1)+1}(g)$$

for all indexes $i = 1, \dots, L$ and all $g \in E_{M,l,x}^k$. Recall that

$$C = \frac{3^{q+1}}{3^{q+1} - 1}, \quad K = \lceil 2 \log_C \frac{16}{\delta} \rceil.$$

Theorem 3.8 applied to the towers \tilde{U}, \tilde{V} now implies that

$$|E_{M,l,x}^k| \leq \left(1 + \frac{\delta}{4 \cdot 3^{q+1}} \right)^{-\lfloor L/K \rfloor} |B(M + l)|,$$

since $\bigcup_{V \in \tilde{\mathcal{V}}} V \subseteq B(M+l)$. It follows that there exist universal constants $c > 0, \rho \in (0, 1)$ such that

$$|E_{M,l,x}^k| \leq c\rho^k |B(M)| + o_{x,l}(|B(M)|)$$

for all k large enough, and the proof is complete.

4.2. Kingman's Subadditive Ergodic Theorem. As one of the applications of Theorem 4.1 we show how the proof of exponential decay in Kingman's subadditive ergodic theorem from [Hoc09] carries over to groups of polynomial growth.

A stochastic process $(S_{B(g,r)})$ on a group of polynomial growth Γ is called *subadditive* if for every ball $V \subset \Gamma$ and for every finite collection of disjoint balls V_1, V_2, \dots, V_n such that $V = \bigcup_{i=1}^n V_i$ we have

$$S_V \leq \sum_{i=1}^n S_{V_i}.$$

Theorem 4.3. *Let $(S_{B(g,r)})$ be a stationary subadditive process on a group of polynomial growth Γ such that for some constant $C > 0$ we have $S_{B(e,0)} \leq C$ almost surely. Then for every interval (α, β) there exist constants $c > 0, \rho \in (0, 1)$, depending only on α, β, Γ and C , such that*

$$\mathbb{P} \left(\left\{ x : \left(\frac{1}{|B(i)|} S_{B(i)} \right)_{i \geq 0} \in \mathcal{F}_{(\alpha, \beta)}^k \right\} \right) \leq c\rho^k$$

for all $k \geq 1$.

Proof. First of all, by stationarity and subadditivity of the process $(S_{B(g,r)})$ we have

$$\frac{1}{|B(g,r)|} S_{B(g,r)}(x) \leq C \quad \text{for all } g \in \Gamma, r \geq 0$$

for all x in a some subset $X_0 \subseteq X$ of full measure. This implies that it suffices to consider only the case $C > \alpha$. Furthermore, by replacing if necessary the interval (α, β) with a smaller subinterval, it suffices to prove the theorem for all (α, β) such that

$$\delta := \frac{\beta - \alpha}{C} < \delta_0,$$

where δ_0 is the 'small enough δ ' given by Theorem 4.1. We apply Theorem 4.1 with this value of δ to the stationary process

$$\left(\frac{1}{|B(g,r)|} S_{B(g,r)} \right),$$

and it only remains to show that the event

$$R := \left\{ x \in X_0 : \begin{array}{l} \exists n > k \text{ s.t. } \frac{1}{|B(n)|} S_{B(n)}(x) > \beta \text{ and } B(n) \text{ can be } \delta\text{-filled} \\ \text{by disjoint balls } V_1, \dots, V_m \text{ s.t. } \forall i \frac{1}{|V_i|} S_{V_i}(x) < \alpha \end{array} \right\}$$

is empty. Suppose the contrary and pick $x \in R$. If a ball $B(n)$ such that $\frac{1}{|B(n)|} S_{B(n)}(x) > \beta$ is δ -filled by the balls V_1, \dots, V_m such that $S_{V_i}(x) < \alpha |V_i|$ for all i , then via subadditivity of the process we obtain the inequality

$$\begin{aligned} \frac{1}{|B(n)|} S_{B(n)}(x) &\leq \sum_{i=1}^m \frac{1}{|B(n)|} S_{V_i}(x) + \frac{1}{|B(n)|} \sum_{g \in B(n) \setminus \bigcup V_i} S_{B(g,0)}(x) \leq \\ &< \sum_{i=1}^m \frac{\alpha |V_i|}{|B(n)|} + \frac{|B(n) \setminus \bigcup V_i|}{|B(n)|} C \leq \alpha + \delta C. \end{aligned}$$

A direct computation shows that $\alpha + \delta C = \beta$, contradiction. \square

Acknowledgements. This research was carried out during the author’s PhD studies in TU Delft under the supervision of Markus Haase. I would like to thank him for his support. The author also kindly acknowledges the financial support from Delft Institute of Applied Mathematics.

REFERENCES

- [Bas72] H. Bass. “The degree of polynomial growth of finitely generated nilpotent groups”. In: *Proc. London Math. Soc. (3)* 25 (1972), pp. 603–614. ISSN: 0024-6115.
- [CSC10] T. Ceccherini-Silberstein and M. Coornaert. *Cellular automata and groups*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2010, pp. xx+439. ISBN: 978-3-642-14033-4. DOI: 10.1007/978-3-642-14034-1. URL: <http://dx.doi.org/10.1007/978-3-642-14034-1>.
- [FS82] G. B. Folland and E. M. Stein. *Hardy spaces on homogeneous groups*. Vol. 28. Mathematical Notes. Princeton University Press, Princeton, N.J. and University of Tokyo Press, Tokyo, 1982, pp. xii+285. ISBN: 0-691-08310-X.
- [Gro81] M. Gromov. “Groups of polynomial growth and expanding maps”. In: *Inst. Hautes Études Sci. Publ. Math.* 53 (1981), pp. 53–73. ISSN: 0073-8301. URL: http://www.numdam.org/item?id=PMIHES_1981__53__5
- [Hoc09] M. Hochman. “Upcrossing inequalities for stationary sequences and applications”. In: *Ann. Probab.* 37.6 (2009), pp. 2135–2149. ISSN: 0091-1798. DOI: 10.1214/09-AOP460. URL: <http://dx.doi.org/10.1214/09-AOP460>.
- [KW99] S. Kalikow and B. Weiss. “Fluctuations of ergodic averages”. In: *Proceedings of the Conference on Probability, Ergodic Theory, and Analysis (Evanston, IL, 1997)*. Vol. 43. 3. 1999, pp. 480–488. URL: <http://projecteuclid.org/euclid.ijm/1255985104>.
- [Nev06] A. Nevo. “Pointwise ergodic theorems for actions of groups”. In: *Handbook of dynamical systems. Vol. 1B*. Elsevier B. V., Amsterdam, 2006, pp. 871–982. DOI: 10.1016/S1874-575X(06)80038-X. URL: [http://dx.doi.org/10.1016/S1874-575X\(06\)80038-X](http://dx.doi.org/10.1016/S1874-575X(06)80038-X).
- [Pan83] P. Pansu. “Croissance des boules et des géodésiques fermées dans les nilvariétés”. In: *Ergodic Theory Dynam. Systems* 3.3 (1983), pp. 415–445. ISSN: 0143-3857. DOI: 10.1017/S0143385700002054. URL: <http://dx.doi.org/10.1017/S0143385700002054>.
- [Mor16] N. Moriakov. “Fluctuations of Ergodic Averages for Actions of Groups of Polynomial Growth”. In: *ArXiv e-prints* (Aug. 2016). arXiv:1608.05033 [math.DS].

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